Lösungen zur 9. Übg Stochastik für Informatiker

Aufg 35

a) $E[aX+bY] = \sum_{x,y} (ax+by)\, P(X=x, Y=y)$

$= \sum_{x,y} ax\, P(X=x, Y=y) + \sum_{x,y} by\, P(X=x, Y=y)$

$= a \sum_x x\, P(X=x) \quad + b \sum_y y\, P(Y=y)$

$= a\, E[X] \quad + b\, E[Y]$

b) $X \leq Y \Rightarrow Y - X \geq 0$

$\Rightarrow E[Y-X] = \left\{ \begin{array}{l} \sum_{x,y} (y-x)\, P(X=x, Y=y) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-x)\, f(x,y)\, dy\, dx \end{array} \right\} \geq 0$

$\overset{\text{Lin.}}{\Rightarrow} E[Y] - E[X] \geq 0 \quad \Rightarrow \quad E[X] \leq E[Y]$

Aufg 34 b)

Setze $T: \begin{cases} (0,\infty)^2 \to (0,\infty)^2 \\ (x_1, x_2) \mapsto \left(\frac{x_1}{x_2}, x_2\right) \end{cases}$

$\Rightarrow T^{-1}(a,b) = (ab, b)$

$\overset{\text{wie a)}}{\Rightarrow}$ T ist bijektiv auf $(0,\infty)^2$

$\det\left(\frac{\partial T_i}{\partial x_j}\right) = \begin{pmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ 0 & 1 \end{pmatrix} = \frac{1}{x_2} \neq 0$

$\Rightarrow \left(\frac{X_1}{X_2}, X_2\right)$ abs. stetig mit Dichte

$f_{\left(\frac{X_1}{X_2}, X_2\right)}(z,t) = t \cdot f_{(X_1,X_2)}(zt, t)\, 1_{(0,\infty)^2}(z,t)$

$\overset{\text{st.u.}}{\Rightarrow} f_{\frac{X_1}{X_2}}(z) = \int_0^{\infty} t\, f_{X_1}(zt)\, f_{X_2}(t)\, dt\; 1_{(0,\infty)}(z)$

Aufg 36

a) $E[X] = \sum_{i=0}^{\infty} i \frac{R^i}{i!} e^{-R} = e^{-R} \sum_{i=1}^{\infty} \frac{R^i}{(i-1)!} = Re^{-R} \sum_{i=0}^{\infty} \frac{R^i}{i!}$

$= Re^{-R} e^{R} = R$

$Var[X] = E[X^2] - (E[X])^2$

$E[X^2] = \sum_{i=0}^{\infty} i^2 \frac{R^i}{i!} e^{-R} = Re^{-R} \sum_{i=1}^{\infty} \frac{iR^{i-1}}{(i-1)!} = Re^{-R} \sum_{i=1}^{\infty} \frac{\frac{d}{dR} R^i}{(i-1)!}$

$= Re^{-R} \frac{d}{dR} \sum_{i=1}^{\infty} \frac{R^i}{(i-1)!} = Re^{-R} \frac{d}{dR} Re^{R} = Re^{-R}(e^{R} + Re^{R})$

$= R(1+R) = R + R^2$

$Var[X] = R + R^2 - R^2 = R$

b) $Bin(n,p) = \underbrace{Geo(p) * \ldots * Geo(p)}_{n}$

$Y_i \sim Geo(p) \quad i = 1, \ldots, n \quad$ st. u.

$X := \sum_{i=1}^{n} Y_i$

$E[X] = \sum_{i=1}^{n} E[Y_i] = n E[Y_1] = \frac{n(1-p)}{p}$

$E[Y_1^2] = E[Y_1(Y_1 - 1) + Y_1] = E[Y_1(Y_1-1)] + E[Y_1]$

$= \sum_{k=0}^{\infty} k(k-1) q^k p + \frac{q}{p}$

$= pq^2 \sum_{k=2}^{\infty} k(k-1) q^{k-2} + \frac{q}{p}$

$= pq^2 \sum_{k=2}^{\infty} \frac{\partial^2}{\partial q^2} q^k + \frac{q}{p}$

$= pq^2 \frac{\partial^2}{\partial q^2} \sum_{k=2}^{\infty} q^k + \frac{q}{p}$

$= pq^2 \frac{\partial^2}{\partial q^2} \frac{q}{1-q} + \frac{q}{p}$

$$= pq^2 \frac{d}{\partial q} \frac{2q(1-q)+q^2}{(1-q)^2} + \frac{q}{p}$$

$$= pq^2 \frac{\partial}{\partial q} \frac{2q-q^2}{(1-q)^2} + \frac{q}{p}$$

$$= pq^2 \frac{(2-2q)(1-q)^2 + 2(1-q)(2q-q^2)}{(1-q)^4} + \frac{q}{p}$$

$$= pq^2 \frac{(2-2q)(1-q) + 2(2q-q^2)}{(1-q)^3} + \frac{q}{p}$$

$$= pq^2 \frac{2}{p^3} + \frac{q}{p} = \frac{2q^2}{p^2} + \frac{q}{p}$$

$$Var[Y_1] = E[Y_1^2] - \left(E[Y_1]\right)^2 = \frac{2q^2+pq-q^2}{p^2} = \frac{q(q+p)}{p^2} = \frac{1-p}{p^2}$$

$$\Rightarrow Var[X] \overset{\text{st.u.}}{=} \sum_{i=1}^{n} Var[Y_i] = n\, Var[Y_1] = \frac{n(1-p)}{p^2}$$

c) $$E[X] = \int_0^\infty x \frac{R^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-Rx}\, dx$$

$$= \frac{R^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-Rx}\, dx$$

$$= \frac{R^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{R^{\alpha+1}} \underbrace{\int_0^\infty \frac{R^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-Rx}\, dx}_{=1}$$

$$= \frac{\alpha}{R}$$

$$E[X^2] = \int_0^\infty x^2 \frac{R^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-Rx}\, dx$$

$$= \frac{R^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{R^{\alpha+2}} \int_0^\infty \frac{R^{\alpha+2}}{\Gamma(\alpha+2)} x^{\alpha+1} e^{-Rx}\, dx = \frac{(\alpha+1)\alpha}{R^2}$$

$$\Rightarrow Var[X] = \frac{(\alpha+1)\alpha}{R^2} - \frac{\alpha^2}{R^2} = \frac{\alpha}{R^2}$$

d) $$E(X^+) = \int_0^\infty \frac{x}{\pi(1+x^2)}\, dx = \lim_{t\to\infty} \int_0^t \frac{x}{\pi(1+x^2)}\, dx = \lim_{t\to\infty} \frac{1}{2\pi} \ln(1+t^2) = \infty \Rightarrow E[X] \nexists$$

$$\Rightarrow Var[X] \nexists$$

Zusatzaufgabe zum 9. Kapitel

Seien $X_1, \dots, X_n$ iid $\sim Poi(\lambda)$
Bestimmen Sie für eine Stichprobe $x_1, \dots, x_n$ den Maximum-Likelihood-Schätzer λ^* von λ.

Lösung:
$$L(\lambda \mid x_1, \dots, x_n) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} = e^{-n\lambda} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\log L(\lambda \mid x_1, \dots, x_n) = -n\lambda + \left(\sum_{i=1}^{n} x_i\right) \log \lambda - \underbrace{\sum_{i=1}^{n} \log x_i!}_{\text{unabh. von } \lambda}$$

$$\Rightarrow \text{maximiere } g(\lambda) = -n\lambda + \left(\sum_{i=1}^{n} x_i\right) \log \lambda$$

notwendiges Kriterium für ein Maximum:
$$g'(\lambda) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} \overset{!}{=} 0$$
$$\Rightarrow \lambda^* = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x}$$

hinreichendes Kriterium:
$$g''(\lambda) = \sum_{i=1}^{n} x_i \left(-\frac{1}{\lambda^2}\right) \overset{!}{<} 0$$
gilt, falls $\sum_{i=1}^{n} x_i > 0$

$\sum_{i=1}^{n} x_i = 0$, $x_i \geq 0$ $\Rightarrow x_i = 0 \;\forall i$

Falls $\sum_{i=1}^{n} x_i = 0$: $\log L(\lambda \mid x_1 = 0, \dots x_n = 0) = -n\lambda$
$\Rightarrow$ Maximum bei $\lambda^* = 0$ $\left(= \frac{\sum x_i}{n}, \text{ da } \sum_{i=1}^{n} x_i = 0\right)$

$\Rightarrow$ Für alle $x_1, \dots, x_n$ ist $\lambda^* = \frac{\sum_{i=1}^{n} x_i}{n}$ ML-Schätzer für λ.